Spotlights in Computational Physics and Engineering (SCoPE)

Invited lectures on:

High-order structure-preserving finite element methods for incompressible flow and elasticity

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Abstract

In the past five to ten years, structure-preserving methods have risen in popularity in the finite element community. With roots in geometric integrators and finite element exterior calculus, structure-preserving methods conserve a particular physical quantity of the underlying system. Typically, one begins with a variational problem posed in function spaces that appear in a differential complex (e.g. the de Rham complex) and chooses discrete spaces that form a (discrete) complex with the same structure. Such a choice will necessarily preserve certain constraints, like Gauss's law in electromagnetism. However, for some problems, like incompressible flow problems, constructing discrete spaces that form a complex is a nontrivial task. Without assumptions on the mesh structure, high-order elements are often required.

In these lectures, we will focus on elements that preserve the divergence constraint arising in incompressible flow and how various properties of these elements arise in both elasticity and fourth-order problems. In particular, we will address the uniform stability properties of the 2D Falk-Neilan and Scott-Vogelius elements in the context of the Stokes equations and how to construct efficient preconditioners. Perhaps surprisingly, these stability properties provide a mathematical explanation for locking in primal discretizations of nearly incompressible linear elasticity. Similar analytical techniques also allow us to show that symmetric $H(\mathrm{div})$-conforming tensor-valued elements that arise in mixed forms of elasticity and Reissner-Mindlin plates are uniformly stable with respect to polynomial degree. Finally, we turn to fourth-order problems where we develop a scheme to compute conforming discretizations using standard software without implementing C^1-continuous elements.

The lectures are structured as follows:

1. Divergence-free elements for Stokes: Review of differential complexes and inf-sup conditions. Falk-Neilan and Scott-Vogelius elements. Constructing inverses of the divergence operator. Optimal error estimates. Preconditioning saddle point problems and augmented Lagrangian preconditioners.
2. Locking in linear elasticity: Robustness for low-order vs. high-order methods. Mesh topology affects robustness. Relation to Scott-Vogelius elements and locking-free estimates. Preconditioning parameter-dependent problems.
3. The elasticity complex and discertizations: Hellinger-Reissner formulation of elasticity. The elasticity complex and boundary conditions. Arnold-Winther and Hu-Zhang elements. Uniform stability. Brief comments on technical tools.
4. Computing conforming solutions for fourth-order problems: Difficulties with C^1 elements, particularly for C^1 splines in 3D. Numerical examples of the vibrating plate problem and convergence tests for 3D elements. Novel mixed form for fourth-order problems and structure-preserving discretizations. Implementation via iterated penalty method.

When and Where?

- 16.01.2025, 10:00-12:00, Maison du Nombre, MNO 1.020
- 17.01.2025, 10:00-12:00, Maison du Nombre, MNO 1.020

Invitee: Charles Parker*

CHARLES PARKER is a postdoc at the University of Oxford funded by the National Science Foundation (U.S.). Previously, he earned his PhD in applied mathematics at Brown University under the supervision of Professor Mark Ainsworth. He is interested in the analysis and implementation of high-order finite element methods with applications to incompressible flow and elasticity.

Selected Publications

- M. AINSWORTH AND **C. PARKER**, *Two and three dimensional H^2-conforming finite element approximations without C^1-elements*, Comput. Methods Appl. Mech. Engrg., 431 (2024), p. 117267. doi:10.1016/j.cma.2024.117267
- M. AINSWORTH AND **C. PARKER**, *Uniform preconditioners for high order finite element approximations of planar linear elasticity*, Math. Comp., 93 (2024), pp. 2067–2102. doi:10.1090/mcom/3926
- M. AINSWORTH AND **C. PARKER**, *Unlocking the secrets of locking: Finite element analysis in planar linear elasticity*, Comput. Methods Appl. Mech. Engrg., 395 (2022), p. 115034. doi:10.1016/j.cma.2022.115034
- M. AINSWORTH AND **C. PARKER**, *A mass conserving mixed hp-FEM scheme for Stokes flow. Part III: Implementation and preconditioning*, SIAM J. Numer. Anal., 60 (2022), pp. 1574–1606. doi:10.1137/21M1433927
- M. AINSWORTH AND **C. PARKER**, *Mass conserving mixed hp-FEM approximations to Stokes flow. Part II: Optimal convergence*, SIAM J. Numer. Anal., 59 (2021), pp. 1245–1272. doi:10.1137/20M1359110
- M. AINSWORTH AND **C. PARKER**, *Mass conserving mixed hp-FEM approximations to Stokes flow. Part II: Uniform stability*, SIAM J. Numer. Anal., 59 (2021), pp. 1218–1244. doi:10.1137/20M1359109
- F. R. A. AZNARAN, K. HU, AND **C. PARKER**, *Uniformly hp-stable elements for the elasticity complex*, sep 2024, `https://arxiv.org/abs/2409.17414`.