Spotlights in Computational Physics and Engineering (SCoPE)

Invited lectures on:

Many facets of cohomology: structure aware formulations and finite element exterior calculus

K. Hu*
kaibo.hu@ed.ac.uk

Abstract

Differential complexes encode important algebraic and differential structures of physics models. Different problems involve different differential structures and complexes. For grad-div-curl related problems, such as those from electromagnetism and fluid dynamics, the de Rham complex plays a fundamental role. For other problems, such as those from continuum mechanics, differential geometry, and general relativity, other complexes are required, such as the so-called elasticity (Kröner, Calabi) complex. These complexes and their properties can be systematically derived from the de Rham complex via a Bernstein–Gelfand–Gelfand (BGG) construction. There appears to be a neat correspondence between a large class of continuum mechanics models and the BGG machinery. Hence, differential complexes also provide a new angle for developing mechanics models and shed light on their structure-aware formulation. In this talk, we discuss the BGG machinery and their correspondence to elasticity, microstructures (micropolar models), continuum defects, dimension reduction, and multi-dimensional models. This paves a way for structure-preserving discretization.

A fundamental question in computational mathematics and computational physics is how to discretize systems that involve multiple variables. Extending finite element or finite difference schemes designed for scalar problems to the Maxwell equations or the Navier–Stokes equations can lead to spurious (i.e., wrong) numerical solutions. This demonstrates the idea of compatible discretization: different variables must satisfy certain compatibility conditions, rather than being chosen arbitrarily.

The principle of identifying such conditions for multi-variable systems is captured by differential complexes (sequences of spaces connected by differential operators) and their cohomologies. Establishing these complexes and cohomological structures at the discrete level enables numerical methods that preserve crucial conserved quantities and constraints. This cohomological approach to compatible and structure-preserving discretization has been developed in the framework of discrete differential forms and Finite Element Exterior Calculus.

Different problems involve different differential structures and complexes. For grad–curl–div-related problems, the de Rham complex plays a fundamental role. For other problems, such as those from continuum mechanics, differential geometry, and general relativity, other complexes are required, such as the so-called elasticity (Kröner, Calabi) complex. These complexes and their properties can be systematically derived from the de Rham complex via a Bernstein–Gelfand–Gelfand (BGG) construction. There appears to be a neat correspondence between a large class of continuum mechanics models and the BGG machinery. Hence, differential complexes also provide a new angle for developing mechanics models and shed light on their structure-aware formulation and structure-preserving discretization.

The lectures are structured as follows:

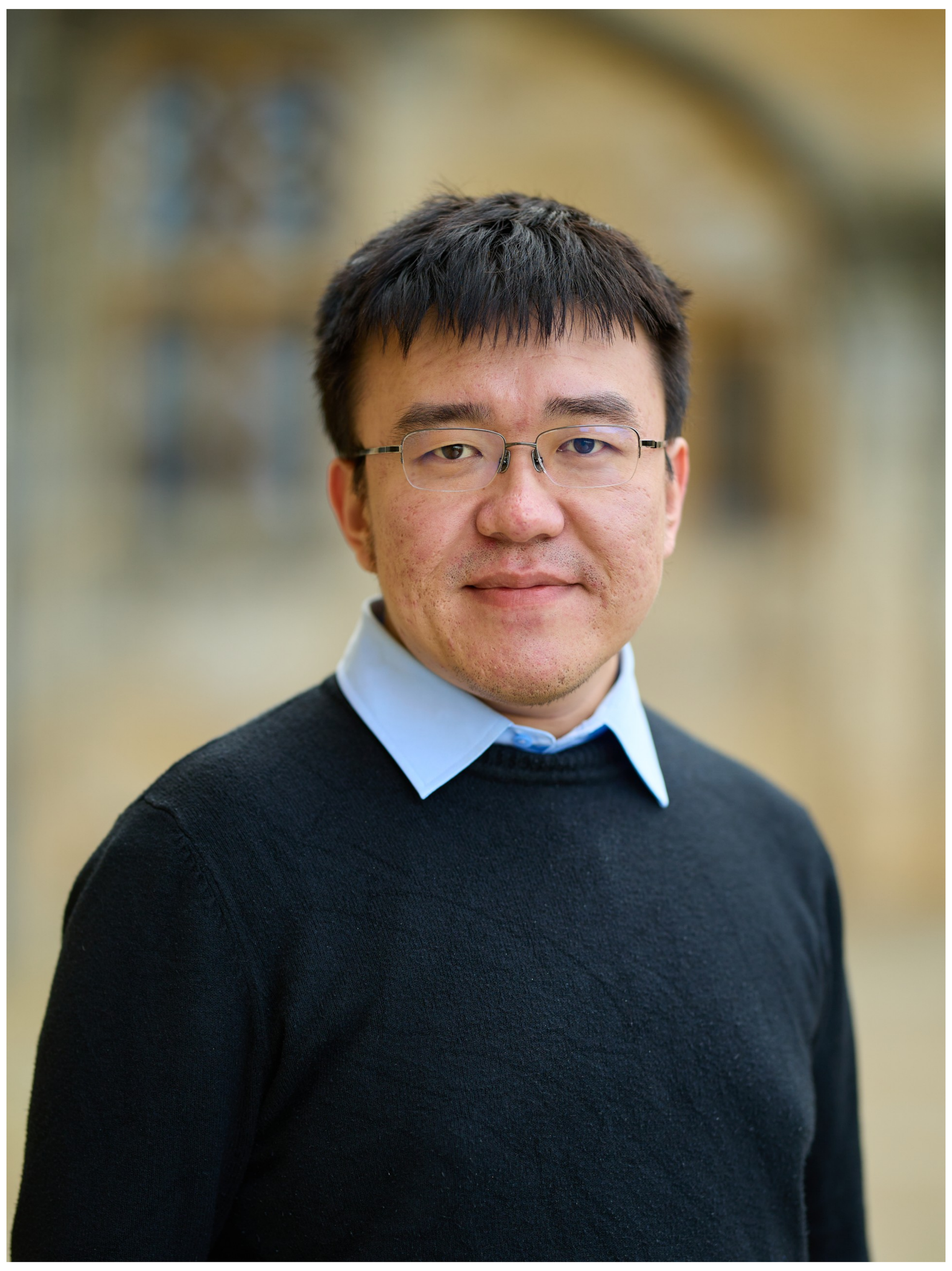
1. Introduction, motivation, and numerical examples. Definition of de Rham complexes and cohomology.
2. The BGG construction as a systematic approach to derive new complexes from the de Rham complex and the associated machinery for continuum mechanics. The correspondence between mechanics models and BGG machinery: elasticity, microstructures, dimension reduction, and multi-dimensional models.
3. Finite Element Exterior Calculus (FEEC): discrete de Rham complexes and discrete differential forms (Lagrange, Nédélec, and Raviart–Thomas elements), and the discretization of BGG complexes.
4. Applications to finite element methods for linear elasticity, Cosserat models and (magneto)hydrodynamics.

When and Where?

- 13.03.2025, 10:00-12:00, Maison du Nombre, MNO 1.030
- 14.03.2025, 10:00-12:00, Maison du Nombre, MNO 1.030

Invitee: Kaibo Hu*

KAIBO HU is a Royal Society University Research Fellow at the Maxwell Institute and School of Mathematics, University of Edinburgh. His research interest includes structure-preserving discretisation, finite element exterior calculus and applications. He received the SIAM Computational Science and Engineering Early Career Prize in 2023 and an ERC Starting Grant “Geometric Finite Element Methods (GeoFEM)” in 2024.



Selected Publications

- Arnold, D. N., Falk, R. S., & Winther, R. (2006). Finite element exterior calculus, homological techniques, and applications. Acta Numerica, 15, 1-155.
- Arnold, D. N., & Hu, K. (2021). Complexes from complexes. Foundations of Computational Mathematics, 21(6), 1739-1774.
- Čap, A., & Hu, K. (2024). BGG sequences with weak regularity and applications. Foundations of Computational Mathematics, 24(4), 1145-1184.
- Christiansen, S. H., Hu, K., & Sande, E. (2020). Poincaré path integrals for elasticity. Journal de Mathématiques Pures et Appliquées, 135, 83-102.
- Dziubek, A., Hu, K., Karow, M., & Neunteufel, M. (2024). Intrinsic mixed finite element methods for linear Cosserat elasticity and couple stress problem. arXiv preprint arXiv:2410.14176.